SOLUTION OF THE ENERGY EQUATION FOR ELECTRIC-ARC HEATING OF A GAS

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There are various approaches to the solution of the energy equation for a cylindrical arc and its application to calculation of the characteristics of electric-arc heating of gases [1–3]. The method expounded in [3] gives numerically accurate results through the use of successive approximations.

In this paper we propose a different approach to the solution of this problem; the heat-conduction function becomes the independent variable and the variable radius becomes the required function. An approximate polynomial of the second degree is used to obtain an approximate solution in finite form. Examples of calculation for air and argon show that this solution is suitable for engineering calculations.

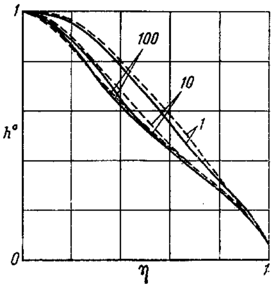

Fig. 1

1. For the case of electric-arc heating of a gas in a cylindrical tube of round section, where the enthalpy h and specific flow rate ρv_z are independent of the longitudinal coordinate z, the energy equation has the form

$$r\frac{\partial s}{\partial r} = -E^2\int_0^r \sigma r\,dr, \qquad s = \int_0^T \lambda dT, \tag{1.1}$$

where λ is the thermal conductivity and the other symbols have their usual meanings. In the conducting region ($s_d^\circ \le s^\circ \le 1$) Eq. (1.1) is put in the form

$$A\eta^2 = -t\int_1^{s^\circ} \sigma^\circ t\,ds^\circ, \tag{1.2}$$

$$\eta = \frac{r}{r_w}, \quad \sigma^\circ = \frac{\sigma}{\sigma_0}, \quad s^\circ = \frac{s}{s_0},$$

$$A = \frac{s_0}{\sigma_0 (Er_w)^2}, \quad t = \eta\frac{d\eta}{ds^\circ}. \tag{1.3}$$

The meanings of the subscripts are: w–wall, 0–tube axis.

In the nonconducting region ($s_w^\circ \le s^\circ \le s_d^\circ$) Eq. (1.1) takes the form

$$\eta ds^\circ / d\eta = -1/B, \tag{1.4}$$

and can be put in the form

$$B\eta^2 = -t,$$

$$B = \frac{2\pi s_0}{EI}, \quad I = 2\pi E\int_0^{r_d} \sigma r dr = \frac{2\pi s_0}{AE}\int_1^{s_d^\circ} \sigma^\circ t\,ds^\circ. \tag{1.5}$$

The last equation can be converted to

$$\int_1^{s_d^\circ} \sigma^\circ t\,ds^\circ = \frac{A}{B}. \tag{1.6}$$

Here I is the total current and the subscript d denotes the boundary of the conducting region. The relationship $\sigma^\circ = \sigma^\circ(s^\circ)$ is assumed to be known. Integrating the last expression of (1.3) with the condition $\eta(1) = 0$, we obtain

$$\eta^2 = 2\int_1^{s^\circ} t ds^\circ. \tag{1.7}$$

We approximate t in the conducting region by a second-degree polynomial

$$t = a_0 + a_1 s^\circ + a_2 s^{\circ 2}. \tag{1.8}$$

To determine the coefficients a_0, a_1, and a_2 we use the boundary values of t and the value of dt/ds° on the tube axis. Thus, Eq. (1.2) and the result of its differentiation are satisfied on the axis, and the integral energy relationship is satisfied in the conducting region.

From (1.2) and (1.7)

$$t = -2A\int_1^{s^\circ} t ds^\circ \left(\int_1^{s^\circ} \sigma^\circ t ds^\circ\right)^{-1}.$$

Hence, expanding the indeterminate form, we find $t(1) = -2A$. On the basis of (1.5) we find $t(s_d^\circ) = -B\eta_d^2$. Differentiating (1.2) twice with respect to s° and using (1.3), we obtain

$$dt/ds^\circ|_{s^\circ=1} = A\alpha, \qquad \alpha = d\sigma^\circ/ds^\circ|_{s^\circ=1}.$$

Proceeding from (1.8) and the obtained boundary values, we find

$$a_0 = m_0 A - n_0 B\eta_d^2, \quad a_1 = -m_1 A + 2n_0 B\eta_d^2,$$
$$a_2 = m_2 A - n_0 B\eta_d^2,$$

where

$$m_0 = \frac{2s_d^\circ(2 - s_d^\circ)}{(1 - s_d^\circ)^2} + \frac{\alpha s_d^\circ}{1 - s_d^\circ},$$

$$m_1 = \frac{4}{(1 - s_d^\circ)^2} + \frac{\alpha(1 + s_d^\circ)}{1 - s_d^\circ},$$

$$m_2 = \frac{2}{(1 - s_d^\circ)^2} + \frac{\alpha}{1 - s_d^\circ},$$

$$n_0 = \frac{1}{(1 - s_d^\circ)^2}.$$

Using (1.8), we obtain

$$\int_1^{s^\circ} \sigma^\circ t ds^\circ = -a_0 I_0 - a_1 I_1 - a_2 I_2,$$

$$I_0 = \int_{s^\circ}^1 \sigma^\circ ds^\circ, \quad I_1 = \int_{s^\circ}^1 \sigma^\circ s^\circ ds^\circ, \quad I_2 = \int_1^{s^\circ} \sigma^\circ s^{\circ 2} ds^\circ,$$

$$\int_1^{s^\circ} t ds^\circ = -a_0(1 - s^\circ) - \frac{a_1(1 - s^{\circ 2})}{2} - \frac{a_2(1 - s^{\circ 3})}{3}.$$

Equation (1.6) takes the form

$$(1 - s_d^\circ)^2(1 - BF)A - (I_{0d} + {}_{2d} - 2I_{1d})B^2\eta_d^2 = 0,$$
$$F = -m_0 I_{0d} + m_1 I_{1d} - m_2 I_{2d}. \tag{1.9}$$

Here I_{0d}, I_{1d}, and I_{2d} are the values of I_0, I_1, and I_2, respectively, when $s^\circ = s_d^\circ$.

Table 1

p^*	1	2	5	10	20	50	100
S_0	2554	2275	2032	1893	1787	1674	1598
σ_0	120.2	103.5	85.11	73.28	57.54	46.77	36.73
S_d°	0.356	0.38	0.3925	0.3930	0.3890	0.3827	0.381
S_w°	0.0094	0.0105	0.0118	0.0127	0.0134	0.0143	0.0150
α	1.902	2.160	2.586	2.841	3.16	3.62	4.057
I_{0d}	0.2796	0.2556	0.2244	0.2052	0.1942	0.1842	0.1779
I_{1d}	0.2262	0.2098	0.1881	0.174	0.1656	0.1575	0.1533
I_{2d}	0.188	0.177	0.1612	0.1507	0.144	0.1375	0.133
B	1.314	1.401	1.532	1.619	1.68	1.74	1.77
A	0.0886	0.0775	0.0611	0.0511	0.0454	0.038	0.0341
Er_w	15.5	16.8	19.8	22.4	25.5	31.0	35.7
I / r_w	789	605	422	327	258	196	158
$10^{-3}\ EI$	12.22	10.20	8.33	7.34	6.73	6.22	5.67
η_d	0.634	0.596	0.558	0.540	0.532	0.526	0.523
Er_w	15.9	17.6	20.3	22.9	25.7	29.9	33.5
I / r_w	732	538	380	298	243	190	165
$10^{-3}\ EI$	11.62	9.30	7.71	6.90	6.30	5.78	5.55
η_d	0.618	0.567	0.533	0.518	0.510	0.511	0.517

Table 2

$10^{-3}\ T_0$	9	10	11	12	13	14
S_0	1135	1750	2722	4180	6140	8615
σ_0	2754	3548	4416	5369	6166	7080
S_d°	0.157	0.1018	0.0655	0.0426	0.029	0.0207
S_w°	0.002	0.0013	0.0008	0.0006	0.0004	0.0003
α	0.7337	0.5992	0.5524	0.4474	0.3875	0.4202
I_{0d}	0.5205	0.5752	0.6198	0.6563	0.684	0.6934
I_{1d}	0.3565	0.3777	0.3935	0.4056	0.4143	0.4151
I_{2d}	0.2668	0.2777	0.2844	0.2906	0.2952	0.298
B	0.7979	0.7324	0.6872	0.6590	0.6391	0.6292
A	0.1778	0.1896	0.1972	0.2037	0.2086	0.2091
Er_w	1.52	1.61	1.77	1.97	2.18	2.41
$10^{-3}\ I / r_w$	5.87	9.31	14.08	20.27	27.63	35.66
$10^{-3}\ EI$	8.94	15.01	24.89	39.85	60.36	86.03
η_d	0.884	0.929	0.956	0.972	0.982	0.987
Er_w	1.72	1.76	1.86	2.03	2.23	2.46
$10^{-3}\ I / r_w$	5.50	9.02	13.65	19.72	27.54	36.31
$10^{-3}\ EI$	9.45	15.88	25.39	40.03	61.41	89.32

When $s^\circ = s_d^\circ$ Eq. (1.7) gives

$$A = \frac{[3-2(1-s_d^\circ)B]\,\eta_d^2}{(1-s_d^\circ)[8+\alpha(1-s_d^\circ)]}. \tag{1.10}$$

Integration of Eq. (1.4) with the condition that $s^\circ(1) = s_w^\circ$ leads to the relationship

$$\eta = \exp[-B(s^\circ - s_w^\circ)]. \tag{1.11}$$

Hence, when $s^\circ = s_d^\circ$ we obtain the equation

$$\eta_d = \exp[-B(s_d^\circ - s_w^\circ)]. \tag{1.12}$$

The system of equations (1.9), (1.10), and (1.12) can be used to determine the values of B, η_d, and A in relation to T_0.

Expressions (1.9) and (1.10) lead to a quadratic equation for B

$$b_0B^2 - b_1B + 3(1-s_d^\circ) = 0,$$

$$b_0 = 2[12+\alpha(2-s_d^\circ - s_d^{\circ 2})]\,I_{1d} - [4(2+2s_d^\circ - s_d^{\circ 2}) + \alpha(1-s_d^\circ)(1+2s_d^\circ)]\,I_{0d} - 3[4+\alpha(1-s_d^\circ)]\,I_{2d},$$

$$b_1 = (1-s_d^\circ)[2(1-s_d^\circ)+3F] = 2(1-s_d^\circ)^2 + \frac{3}{1-s_d^\circ}\{[4+\alpha(1-s_d^{\circ 2})]\,I_{1d} - s_d^\circ[2(2-s_d^\circ+\alpha(1-s_d^\circ)]\,I_{0d} - [2+\alpha(1-s_d^\circ)]\,I_{2d}\}. \tag{1.13}$$

It is obvious that

$$I_{0d} + I_{2d} - 2I_{1d} = \int_{s_d^\circ}^{1}(1-s^\circ)^2\sigma^\circ\,ds^\circ > 0. \tag{1.14}$$

The function

$$f(s^\circ) = m_0 - ms^\circ - m_2s^{\circ 2}$$

at the ends of the interval $(s_d^\circ, 1)$ takes the values $f(s_d^\circ) = 0$, $f(1) = -2$ and in this interval has a single extremum

$$f\left(\frac{m_1}{2m_2}\right) = \frac{4m_0m_2 - m_1^2}{4m_2} = -\frac{[4+\alpha(1-s_d^\circ)]^2}{4[2+\alpha(1-s_d^\circ)]} < 0.$$

Hence, in this interval $f(s^\circ) \le 0$ and

$$F = -\int_{s_d^\circ}^{1}\sigma^\circ f(s^\circ)\,ds^\circ > 0, \tag{1.15}$$

and, hence, $b_1 > 0$.

Since $A > 0$, then from Eq. (1.9), using (1.14) and (1.15), we obtain

$$B < \frac{1}{F}, \qquad B = \frac{b_1 \pm \sqrt{b_1^2 - 12(1-s_d^\circ)b_0}}{2b_0}. \tag{1.16}$$

Here B is given by the quadratic equation (1.13) and we take the root which satisfies condition (1.16).

We then determine the values of η_d and A from (1.12) and (1.10). The characteristics EI, Er_w, and I/r_w are expressed in terms of A and B

$$EI = \frac{2\pi s_0}{B}, \quad Er_w = \left(\frac{s_0}{A\sigma_0}\right)^{1/2}, \quad \frac{I}{r_w} = \frac{2\pi\sqrt{A\sigma_0 s_0}}{B}.$$

The enthalpy characteristic $h^\circ = h^\circ(\eta)$, $h^\circ = h/h_0$, p = const is determined from (1.7), (1.8), and (1.11) with the aid of the relationship $h = h(s, p)$, which is assumed to be known.

2. Calculations for air and argon were made. For air the calculations were carried out for $T_0 = 6000°$ K, $T_d = 4000°$ K, $T_w = 780°$ K and pressures $p = (1\text{-}100)\cdot 1.01325\cdot 10^5$ N/m². We used the relationship $\sigma = \sigma(T, p)$ from [4] and the relationship $\lambda = \lambda(T, p)$ from the data of R. M. Sevost'yanov and M. D. Zdunkevich. Here and henceforth we use the International System of Units (SI).

The data for the calculation and the obtained values of B, A, η_d, Er_w, I/r_w, and EI for different values of $p^* = 10^{-5}\cdot p/1.01325$ are given in Table 1; for comparison the last four rows of the table give the values of η_d, Er_w, I/r_w, and EI obtained numerically by successive approximations from Eq. (1.1) brought to integral form. The numerical method of solution used here, which is similar to the known method [3, 6], was supplemented by a determination of the radius of the conducting region. As an initial approximation we approximated the relationship $\sigma^\circ = \sigma^\circ(\eta)$ by a fourth-degree polynomial.

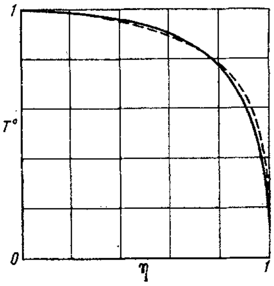

Fig. 2

A comparison of the distributions $h^\circ = h^\circ(\eta, p)$, found from known $s^\circ = s^\circ(\eta, p)$ by means of tables [5], is made in Fig. 1. Here and in Fig. 2 the dashed lines correspond to the proposed approximate method.

The calculations for argon were made with $p = 1.01325\cdot 10^5$ N/m², $T_0 = 9000\text{–}14\,000°$ K, $T_w = 1000°$ K, $T_d = 5500°$ K.

Table 2 gives the data for the calculations and the obtained results; for comparison the four last rows of the table give the values of Er_w, I/r_w, and EI from the results of [6] with the use of additional data on the variation of I/r_w with T_0 in the absence of radiation.

The difference in the values of η_d, Er_w, I/r_w, and EI for air does not exceed 5.1, 6.7, 12.4, and 9.7%, respectively. For argon, the difference in the values of Er_w, I/r_w, and EI does not exceed 11.6, 6.8, and 5.4%, respectively.

Figure 2 compares the results of calculation of the dimensionless distribution of temperature $T^\circ = T/T_0$ by our approximate method and from [6].

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